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Poisson-Lie T-Duality: the Path-Integral Derivation

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Abstract

We formulate Poisson-Lie T-duality in a path-integral manner that allows us to analyze the quantum corrections. Using the path-integral, we rederive the most general form of a Poisson-Lie dualizeable background and the generalized Buscher transformation rules it has to satisfy.

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1 Introduction

The motive that fuels diverse research effort in the field of duality is the hope that the "great unification" of String Theory and low-energy physics exists. Dualities (of different kinds) are a product of String Theory. As such, ultimately they should be able to allow us to link the properties of space-time with the properties of the particle spectrum through the appropriate choice of compactification.

Speaking less ambitiously, target space dualities provide us with a prescription how to identify apparently different target space backgrounds as equivalent or, to put it differently, how to classify physically inequivalent string vacua and, thus, access the moduli space of String Theory.

Dualities relate different regimes of the theory. Target space duality was originally discovered in toroidal compactifications where it changed our understanding of the concept of distance. Namely, as it turns out, there is a minimal (self-dual) distance $R_0 = R_0^{-1}$ in String Theory. Any smaller distance $R < R_0$ is equivalent to $R' = R^{-1} > R_0$.

Recently, Klimčik and Ševera proposed a generalization of abelian and traditional non-abelian dualities called Poisson-Lie T-duality [1]. They classified all Poisson-Lie dualizeable backgrounds and proved the phase space equivalence of a model and its Poisson-Lie dual. Later, a path-integral argument was given [2] for the very special case when the σ-model admits chiral equations of motion. We present a completely general, unambiguous path-integral scheme that allows one to handle Poisson-Lie T-duality and that reduces to the established ways of treating "older" target space dualities.

As duality research progressed towards complicated setups yielding sometimes quite unexpected results, it became more and more important to clearly realize what does one mean by "equivalence" in duality.

In the simplest case, backgrounds with abelian isometries (toroidally compactified theories) were proven to be equivalent Conformal Field Theories, at least in the lowest order of the string coupling α'. Traditional non-abelian duals were proven to be conformal (except for non-semisimple duality groups) [6, 7], with the relation between those CFTs as that of an orbifold by a continuous group [5].

As we intend to demonstrate in Section 4, non-abelian dual backgrounds (in both traditional and Poisson-Lie T-dualities) are equivalent in the same

sense as in the case of abelian duality modulo global issues[2] [8] (except for a contribution to the trace anomaly if the structure constants are not traceless [5, 6, 7]). Our path-integral procedure also shows that the dual models have to incorporate certain quantum corrections of their respective classical target space backgrounds (dilaton shifts in both models as opposed to only one of them).

The outline of the paper is as follows. We start by explaining the original (algebraic) framework of Poisson-Lie target space duality due to Klimčik and Ševera [1] that states that dualizeable backgrounds must satisfy a certain system of partial differential equations (2.9), (2.11) and an extra condition (2.12). Next, in Section 3 we construct the most general background that solves (2.9, 2.11, 2.12). Using the result of Section 3, we introduce the new path-integral formulation (4.2) of Poisson-Lie T-duality in Section 4. We perform the analysis of quantum corrections in the path-integral in Section 5 and incorporate spectator fields in the Section 6 for completeness. We use the actions we obtained with our method to derive the generalized Buscher rules (6.8) for Poisson-Lie duality. We also make thorough comparison of all of our results with the abelian and traditional non-abelian dualities.

2 Basics of Poisson-Lie T-Duality

In this section we introduce the scheme of Poisson-Lie T-duality which was recently proposed by Klimčik and Ševera [1]. Within this new formalism, σ-models dual to each other are treated on equal footing and the existence of inverse duality transformations (which was lacking in "traditional" non-abelian duality [4, 5]) is no longer a problem.

The central idea of [1] was to abandon the requirement that a σ-model has a set of isometries (and corresponding to them conservation laws) in favor of the so-called "non-commutative conservation laws" which can be thought of as non-abelian generalization of the usual conservation laws.

Suppose we are given a σ-model defined on some manifold $\mathcal{M}$:

$$\mathcal{S}[x] = \int d^2z\, \partial x^i F_{ij}(x) \bar{\partial} x^j \tag{2.1}$$

[2]We want to stress that global issues are not addressed in this paper. This paper achieves for Poisson-Lie T-duality what the second paper of [9] achieved for abelian T-duality.

We can define the left (right) group action as

$$\delta x^i = \varepsilon^a v_a^i, \tag{2.2}$$

where ${v^i}_a$ are correspondingly right (left) invariant frames in the Lie algebra $\mathcal{G}$ of the group G. The reasoning is as follows:

$$\delta g = \varepsilon g \Longrightarrow \varepsilon = \delta g g^{-1} = e_i^R \delta x^i \quad \text{or} \quad \delta x^i = \varepsilon^a v_{R\,a}^i, \quad v = e^{-1}. \tag{2.3}$$

This variation generates the currents[3]

$$K_a = \partial x^i F_{ij} v_a^j \qquad \bar{K}_a = v_a^i F_{ij} \bar{\partial} x^j \tag{2.4}$$

and the current 1-form

$$\mathcal{K}_a = K_a dz + \bar{K}_a d\bar{z}. \tag{2.5}$$

The total variation of (2.1) under (2.2) equals:

$$\delta \mathcal{S} = \int d\varepsilon^a \wedge \star \mathcal{K}_a + \int d^2 z \, \partial x^i \bar{\partial} x^j \varepsilon^a \mathcal{L}_{v_a} F_{ij} \, , \tag{2.6}$$

where $\mathcal{L}$ denotes a Lie derivative, and $\star$ is a Hodge star.

We now say that the given model (2.1) admits "non-commutative conservation laws" if the following equation[4] holds on-shell:

$$d \star \mathcal{K}_a + \frac{1}{2} \tilde{f}_a^{bc} \star \mathcal{K}_b \wedge \star \mathcal{K}_c = 0. \tag{2.7}$$

The above equations of motion are automatically satisfied if $\star\mathcal{K}$ is a pure gauge field in certain "dual" algebra $\tilde{\mathcal{G}}$:

$$\star \mathcal{K}_a \tilde{T}^a \equiv \star \mathcal{K} = \tilde{g}^{-1} d\tilde{g} \in \tilde{\mathcal{G}}, \qquad \tilde{g} \in \tilde{G}, \qquad \left[\tilde{T}^b, \tilde{T}^c\right] = \tilde{f}_a^{bc} \tilde{T}^a \tag{2.8}$$

This is the general feature of duality: equation of motion becomes a Bianchi identity when expressed in terms of dual fields.

Note that we have called (2.7) "equations of motion": we have already implied that we consider only σ-models with the following property of their backgrounds F_{ij} (c.f. (2.4)):

$$\mathcal{L}_{v_a} F_{ij} = F_{ik} v_b^k \tilde{f}_a^{bc} v_c^l F_{lj}. \tag{2.9}$$

[3]When F_{ij} has a Wess-Zumino term, there may be corrections to (2.4) as discussed in [3].

[4]Note that the usual conservation law is simply $d \star \mathcal{K} = 0$ which corresponds to $\tilde{f} = 0$ in (2.7).

Equation (2.9) is the central statement of the Klimčik-Ševera approach, because it is a geometrical, non-dynamical equation for the background field F_{ij}. By demanding the closure of (2.9) ($[\mathcal{L}_{v_a}, \mathcal{L}_{v_b}] = f^c_{ab}\mathcal{L}_{v_c}$), we obtain the following consistency condition:

$$f^a_{dc}\tilde{f}^{rs}_a = \tilde{f}^{as}_c f^r_{da} + \tilde{f}^{ra}_c f^s_{da} - \tilde{f}^{as}_d f^r_{ca} - \tilde{f}^{ra}_d f^s_{ca}, \tag{2.10}$$

which is known in mathematics to be the relation for the structure constants of the Lie bi-algebra $(\mathcal{G}, \tilde{\mathcal{G}})$ (for a review, see e.g. [10]).

It is now natural to postulate that the dual to (2.1) σ-model should obey the same condition as (2.9) but with the tilded and un-tilded variables interchanged:

$$\mathcal{L}_{\tilde{v}^a}\tilde{F}^{ij} = \tilde{F}^{ik}\tilde{v}^b_k f^a_{bc}\tilde{v}^c_l \tilde{F}^{lj}. \tag{2.11}$$

and the backgrounds related by

$$(F(x=0))^{-1}_{ij} = \tilde{F}^{ij}(\tilde{x}=0). \tag{2.12}$$

It should be noted that the dilaton contribution cannot be estimated within this approach [1].

As we have mentioned above, a new group structure, called Drinfeld double, emerges in Poisson-Lie T-duality. The Drinfeld double D comes equipped with an inner product (non-degenerate symmetric bi-linear form) that is invariant under the adjoint action of the full double, and has the following properties:

$$\langle T_a, T_b\rangle = 0 \qquad \langle\tilde{T}^a, \tilde{T}^b\rangle = 0 \qquad \langle T_a, \tilde{T}^b\rangle = \delta^b_a\,, \tag{2.13}$$

$$\langle l\, T^A l^{-1}, T^B\rangle = \langle T^A, l^{-1}T^B l\rangle \qquad l \in D, \tag{2.14}$$

where $T_a \in \mathcal{G}$, $\tilde{T}^a \in \tilde{\mathcal{G}}$ and $T^A \in \mathcal{D}$. Also note that any $l \in D$ can be uniquely decomposed (at least close to the identity):

$$l = \tilde{h}g = h\tilde{g} \qquad \text{where } g, h \in G \quad \tilde{g}, \tilde{h} \in \tilde{G} \tag{2.15}$$

Let us also introduce some notation used later in the paper:

$$\begin{aligned} \mu^{ab}(g) &= \langle g\,\tilde{T}^a\,g^{-1}, \tilde{T}^b\rangle \qquad \nu^a_b(g) = \langle g\,\tilde{T}^a\,g^{-1}, T_b\rangle \\ \alpha^a_b(\tilde{g}) &= \langle \tilde{g}\,T_b\,\tilde{g}^{-1}, \tilde{T}^a\rangle \qquad \beta_{ab}(\tilde{g}) = \langle \tilde{g}\,T_a\,\tilde{g}^{-1}, T_b\rangle \end{aligned} \tag{2.16}$$

One may check that $\nu(g^{-1}) = \nu^{-1}(g)$, $\alpha(\tilde{g}^{-1}) = \alpha^{-1}(\tilde{g})$, $\mu(g^{-1}) = \mu^t(g)$ and $\beta(\tilde{g}^{-1}) = \beta^t(\tilde{g})$ where t stands for transpose.

This completes our overview of the algebraic approach to Poisson-Lie target space duality. Our next step is to find a general solution of (2.9) and then, using it as a guide, find a path-integral formulation of Poisson-Lie target space duality.

3 General Solution to Poisson-Lie T-Duality

For simplicity, in this section we consider duality without spectator fields. In other words, the target space can be identified with the group manifold of G (or $\tilde{G}$). The full problem will be treated in section 6.

In traditional non-abelian duality one starts from a σ-model

$$\mathcal{S}_{\mathrm{TNAD}} = \int d^2z\, (g^{-1}\partial g)^a E_{ab} (g^{-1}\bar{\partial} g)^b, \tag{3.1}$$

where E_{ab} is some constant matrix. Comparing this with (2.1) and (2.3) one can see that

$$E_{ab} = v^i_{La} F_{ik} v^k_{Lb} \tag{3.2}$$

is the construct one should work with in order to be able to make a direct connection with traditional non-abelian duality. However, the equation that was solved in [1] is

$$\mathcal{L}_{v_{Rc}}(F_{ij}) = F_{ik} v^k_{Ra} \tilde{f}^{ab}_c v^l_{Rb} F_{lj}, \tag{3.3}$$

and the solution given in [1] is:

$$v^i_{Ra} F_{ik} v^k_{Rb} = \left((\nu^{-1})^a_c + E^0_{cd}\mu^{da} \right)^{-1} E^0_{ce} \nu^e_b, \tag{3.4}$$

where E^0_{ab} is some constant matrix.

We can get the background we are interested in by multiplying the Klimčik-Ševera solution on both sides with

$$e^a_{Ri} v^i_{Lb} = \langle g T_b g^{-1}, \tilde{T}^a \rangle = \nu^{-1}, \tag{3.5}$$

which yields

$$E_{ab}(g) \equiv v^i_{La} F_{ik} v^k_{Lb} = \left(\left(E^0_{ab} \right)^{-1} + \mu^{ac} \nu^b_c \right)^{-1}. \tag{3.6}$$

Similarly the dual background is given by

$$\tilde{E}^{ab}(\tilde{g}) \equiv \tilde{v}_L^{ai}\tilde{F}_{ik}\tilde{v}_L^{bk} = \left(E^0_{ab} + \beta_{ac}\alpha^c_b\right)^{-1}. \tag{3.7}$$

Note that the matrices $\mu\nu$ and $\beta\alpha$ are always *anti-symmetric*[5], in particular this means that this duality relates backgrounds without torsion to backgrounds with torsion.

We see that if the dual group is abelian with coordinates χ_a, (3.6) and (3.7) reduces to

$$E_{ab}(g) = E^0_{ab}, \tag{3.8}$$

$$\tilde{E}^{ab}(\tilde{g}) = \left(E^0_{ab} + \chi_c f^c_{ab}\right)^{-1}, \tag{3.9}$$

which is compatible with traditional non-abelian duality [4, 5].

4 Path-Integral Construction of Poisson-Lie T-Duality

Using the inner product $\langle , \rangle$ one can write a WZW model defined on a Drinfeld double. Since the invariant form is identically zero on each of the sub-algebras $\mathcal{G}$ and $\tilde{\mathcal{G}}$, a WZW model defined through $\langle , \rangle$ will vanish on these and therefore the Polyakov-Wiegmann identity will have a very simple form if applied to $I[g\tilde{h}]$ where $g \in G$ and $\tilde{h} \in \tilde{G}$

$$\begin{aligned} I[g\tilde{h}] &= I[g] + I[\tilde{h}] + \int d^2z\, \langle g^{-1}\partial g, \bar{\partial}\tilde{h}\tilde{h}^{-1}\rangle \\ &= \int d^2z\, \langle g^{-1}\partial g, \bar{\partial}\tilde{h}\tilde{h}^{-1}\rangle. \end{aligned} \tag{4.1}$$

We are going to show how Poisson-Lie T-duality can be derived from a constrained WZW-model on the double ($l \in D$):

$$\mathcal{Z} = \int \mathcal{D}l\, \delta\left[\langle l^{-1}\partial l, \tilde{T}^a\rangle E^0_{ab} - \langle l^{-1}\partial l, T_b\rangle\right] \exp(-I[l]). \tag{4.2}$$

While the above choice may seem strange and arbitrary, let us point out that it is classically equivalent to another well-known model: a WZW model with current-current interaction:

$$\mathcal{S}\left[l_1, l_2\right] = I\left[l_1 l_2\right] - \int d^2z\, \langle l_1^{-1}\partial l_1, T^A\rangle E^0_{AB}\langle T^B, \bar{\partial} l_2 l_2^{-1}\rangle\,, \tag{4.3}$$

[5]This follows from $\langle \mathcal{A}, \mathcal{B}\rangle = \langle \mathcal{A}, T_a\rangle\langle \tilde{T}^a, \mathcal{B}\rangle + \langle \mathcal{A}, \tilde{T}^a\rangle\langle T_a, \mathcal{B}\rangle$.

where E^0_{AB} is a block-diagonal constant matrix such, that $\left(E^0_{ab}\right)^{-1} = E^{0\,\tilde{a}\tilde{b}}$ and $E^0{}_a{}^{\tilde{b}} = E^{0\,\tilde{a}}{}_b = 0$.

To obtain (4.2) one should first truncate l_2 to lie in G (or $\tilde{G}$). Then, after using the Polyakov-Wiegmann identity (4.1) to decompose $I[l_1 l_2]$, one finds that $\bar{\partial} l_2 l_2^{-1}$ becomes a Lagrange multiplier enforcing the delta-function constraint of (4.2).

Unfortunately, in transition from (4.3) to (4.2) one acquires extra Jacobians that complicate further analysis so we will only use this observation to find the starting point for the dualization process.

If we decompose the group element l in (4.2) as $l = \tilde{h}g$, the path integral can be re-written as

$$\int \mathcal{D}(\tilde{h}g)\delta\left[g^{-1}\partial g E^0 + \tilde{h}^{-1}\partial\tilde{h}\left(\mu^t E^0 - \nu^{-1}\right)\right] \times \exp\left(-\int d^2z\, \langle \tilde{h}^{-1}\partial\tilde{h}, \bar{\partial} g g^{-1}\rangle\right). \tag{4.4}$$

Since the left-invariant Haar measure splits into

$$\mathcal{D}(\tilde{h}g) = \mathcal{D}\tilde{h}\mathcal{D}g \det\left(\nu^{-1}\right), \tag{4.5}$$

we obtain, introducing the notation $J = g^{-1}\partial g$, $\tilde{A} = \tilde{h}^{-1}\partial\tilde{h}$

$$\int \mathcal{D}g\mathcal{D}\tilde{h} \det\left(\left(E^0\right)^{-1} E(g)\right)\delta\left[\tilde{A} - JE\nu\right]\exp\left(-\int d^2z\, \tilde{A}\nu^{-1}\bar{J}\right). \tag{4.6}$$

If we instead decompose the group element l as $l = h\tilde{g}$ and perform the analogous calculation, we get

$$\int \mathcal{D}h\mathcal{D}\tilde{g} \det\left(\tilde{E}(\tilde{g})\right)\delta\left[A - \tilde{J}\tilde{E}\alpha\right]\exp\left(-\int d^2z\, A\alpha^{-1}\bar{\tilde{J}}\right), \tag{4.7}$$

where now $\tilde{J} = \tilde{g}^{-1}\partial\tilde{g}$, $A = h^{-1}\partial h$.

Integration over $\tilde{h}$ in (4.6) or h in (4.7) produces σ-models with backgrounds given by (3.6) and (3.7) respectively. Integration also produces non-trivial Jacobians from changing variables, which will be the topic of the next section.

It is interesting to investigate what happens to (4.2) in the case when $\tilde{G}$ is abelian. If we decompose l as $l = h\tilde{g}$ and introduce a lagrange multiplier $\bar{A}$ to put the delta function in the action we get

$$S = \int d^2z\left(A\bar{\tilde{J}} + \left(A\alpha^{-1}E^0 - \tilde{J} - A\beta^t\right)\bar{A}\right). \tag{4.8}$$

Now, using that in the abelian case the formulas (2.16) simplify to (χ are the coordinates on the abelian group $\tilde{G}$)

$$\alpha = \mathbf{1} \quad \beta = \chi_c f^c_{ab} \equiv \chi_c f^c, \tag{4.9}$$

and that the (abelian) currents can be written as $\tilde{J} = \partial\chi$, we can write the action as

$$S = \int d^2z \left(A\bar{\partial}\chi - \partial\chi\bar{A} + A\left(E^0 + \chi_c f^c\right)\bar{A}\right). \tag{4.10}$$

By partial integration we can transform this into two terms; a constraint saying that the field strength of $A, \bar{A}$ should vanish (χ becomes the lagrange multiplier) and a term $AE_0\bar{A}$. Thus we recover the formulation of traditional non-abelian duality in the form it has after gauge fixing the coordinates on G to zero. In our formalism, the "gauge field" $A, \bar{A}$, which is not a gauge field since we recover the traditional formalism *after* gauge fixing, comes from combining a lagrange multiplier and a field coming from the decomposition of l whereas in the traditional case the $A, \bar{A}$ field comes from gauging the isometries of the background E_0. The fact that β and J were related to χ in a simple manner was also important for this equivalence to work. This discussion might give some hints as to why (usually) the dual background have no isometries. Only in the very special case described above do different components combine to give us the usual "gauging of isometries" description.

5 Quantum Analysis

The determinants in (4.6) and (4.7) can be computed using standard heat kernel regularization techniques [9, 11]. The result can be absorbed in the extra shift of the dilaton

$$\Phi' = \Phi^0 + \ln\left(\det\left(E(g)\right)\right) - \ln\left(\det\left(E^0\right)\right) \tag{5.1}$$

and for the dual model

$$\Phi' = \Phi^0 + \ln\left(\det\left(\tilde{E}(\tilde{g})\right)\right) \tag{5.2}$$

Observe that these factors are normalized so that when $\tilde{G}$ is abelian, which means that $E(g) = E^0$, the dilaton shift (5.1) is zero and the shift (5.2) is the same as in the traditional non-abelian duality case. Our result thus reproduces the result of traditional non-abelian duality. However, when both

groups are non-abelian, one has to shift the dilaton in both models to maintain conformal invariance.

To be able to integrate out A (or $\tilde{A}$) we have to change variables in the path integral as follows:

$$\mathcal{D}\tilde{h} = \mathcal{D}\tilde{A}\left(\det(\partial + [\tilde{A}, \cdot])\right)^{-1} \tag{5.3}$$

or

$$\mathcal{D}h = \mathcal{D}A\left(\det(\partial + [A, \cdot])\right)^{-1}. \tag{5.4}$$

Let us analyze what happens to (4.7) using (5.4) (the other case being totally analogous). To avoid gravitational anomalies and not to introduce extra contributions to the central charge, we see that we have to include an extra factor of $\det(\partial)$ in our starting path-integral (4.2). If we do this, the factor that we need to worry about is

$$\int \mathcal{D}A \frac{\det(\partial)}{\det(\partial + [A, \cdot])}\delta\left[A - \tilde{J}\tilde{E}\alpha\right]. \tag{5.5}$$

To make it more symmetrical, we integrate also over $\bar{A}$ and include an additional delta function:

$$\int \mathcal{D}A\mathcal{D}\bar{A} \frac{\det(\partial)\det(\bar{\partial})}{\det(\partial + [A, \cdot])\det(\bar{\partial} + [\bar{A}, \cdot])}\delta\left[A - \tilde{J}\tilde{E}\alpha\right]\delta\left[\bar{A}\right]. \tag{5.6}$$

The determinants in the denominator can be lifted into the action using two extra scalar fields $\phi, \phi^\star$ transforming in the adjoint and coadjoint representations of G respectively. We thus acquire a factor depending on $A, \bar{A}$ in the path-integral

$$N(A, \bar{A}) = \int \mathcal{D}\phi^\star\mathcal{D}\phi \exp\left(-\frac{1}{2\pi}\int d^2z\, \phi^\star\nabla^\mu\nabla_\mu\phi\right). \tag{5.7}$$

To investigate the conformal properties of N we couple it to the world sheet gravity and calculate the one-loop effective action. The purely gravitational parts gives a contribution to the central charge which is cancelled by the compensating factors $\det\left(\partial\bar{\partial}\right)$ introduced above. There is also a mixed anomaly, with the fluctuation of the metric and a "gauge field" as external fields. It gives a contribution to the trace of the energy momentum tensor proportional to the trace of the structure constants which cannot be absorbed in the dilaton, exactly as in [5, 6]. This means that if $f^a_{ab} \neq 0$, the $\tilde{G}$ model

is not conformal, and conversely if $\tilde{f}_a^{ab} \neq 0$, the G model is not conformal. One may also check, using heat-kernel methods, that there are no additional logarithmic divergences in N.

Our result, depending on an "extra" finite, conformally invariant factor, is the generalization of what one encounters in ordinary abelian duality [9]. In principle $N(A, \bar{A})$ could contribute to higher order in the effective action, but this issue will not be pursued further here.

6 Spectators

We now return to the full problem as promised. The general σ-model can be written as

$$\mathcal{S} = \int d^2z \left(\partial x^i F_{ik} \bar{\partial} x^k + \partial x^i F^R_{i\alpha} \bar{\partial} x^\alpha + \partial x^\alpha F^L_{\alpha i} \bar{\partial} x^i + \partial x^\alpha F_{\alpha\beta} \bar{\partial} x^\beta\right), \quad (6.1)$$

where greek indices $\alpha, \beta \ldots$ are associated with inert coordinates. The currents generated by the left group action (2.2) are

$$\begin{aligned} K_a &= \left(\partial x^i F_{ik} + \partial x^\alpha F^L_{\alpha k}\right) v^k_{Ra}, \\ \bar{K}_a &= v^k_{Ra} \left(F_{ki} \bar{\partial} x^i + F^R_{k\alpha} \bar{\partial} x^\alpha\right). \end{aligned} \quad (6.2)$$

If we want them to obey non-commutative conservation laws in the same way the currents (2.4) do, we also have to let F^L, F^R and $F_{\alpha\beta}$ transform under the left action of the group G. The following equations (similar to (2.9)) should be satisfied:

$$\begin{aligned} \mathcal{L}_{v_{Rc}}(F_{ik}) &= F_{ij} v^j_{Ra} \tilde{f}^{ab}_c v^l_{Rb} F_{lk}, \qquad & \mathcal{L}_{v_{Rc}}(F^L_{\alpha i}) &= F^L_{\alpha k} v^k_{Ra} \tilde{f}^{ab}_c v^l_{Rb} F_{li}, \quad (6.3) \\ \mathcal{L}_{v_{Rc}}(F^R_{i\alpha}) &= F_{ik} v^k_{Ra} \tilde{f}^{ab}_c v^l_{Rb} F^R_{l\alpha}, \qquad & \mathcal{L}_{v_{Rc}}(F_{\alpha\beta}) &= F^L_{\alpha i} v^i_{Ra} \tilde{f}^{ab}_c v^k_{Rb} F^R_{k\beta}. \end{aligned}$$

To get the full solution, we use (4.3) as a guide: let E^0 depend on the spectator coordinates and include additional matrices $F^L_{\alpha a}(x^\alpha)$, $F^R_{a\alpha}(x^\alpha)$, $F_{\alpha\beta}(x^\alpha)$ that couple to the spectator coordinates in a natural way:

$$\begin{aligned} \mathcal{S} &= I[l_1 l_2] - \int d^2z \, \Big[\langle l_1^{-1} \partial l_1, \tilde{T}^a \rangle E^0_{ab} \langle \tilde{T}^b, \bar{\partial} l_2 l_2^{-1} \rangle \quad (6.4) \\ &- \langle l_1^{-1} \partial l_1, \tilde{T}^a \rangle F^R_{a\alpha} \bar{\partial} x^\alpha + \partial x^\alpha F^L_{\alpha b} \langle \tilde{T}^b, \bar{\partial} l_2 l_2^{-1} \rangle - \partial x^\alpha F_{\alpha\beta} \bar{\partial} x^\beta \Big] \end{aligned}$$

Upon integrating over $l_2 \in G$ (ignoring Jacobians in analogy to the procedure followed in section 4), we obtain

$$\begin{aligned} \mathcal{Z} &= \int \mathcal{D}l \det(\partial)\ \delta\left[\langle l^{-1}\partial l, \tilde{T}^b\rangle E^0_{ba} + \partial x^\alpha F^L_{\alpha a} - \langle l^{-1}\partial l, \tilde{T}_a\rangle\right] \times \\ & \exp\left[-I[l] - \int d^2z\ \left(\langle l^{-1}\partial l, \tilde{T}^a\rangle F^R_{a\alpha}\bar{\partial}x^\alpha + \partial x^\alpha F_{\alpha\beta}\bar{\partial}x^\beta\right)\right], \end{aligned} \tag{6.5}$$

where we have included the extra $\det(\partial)$ discussed in the previous section.

We can now take (6.5) as our starting point for the dualization process and, by repeating the same arguments, we end up with an action defined on G:

$$\begin{aligned} \mathcal{S} &= \int d^2z\ \left[JE\bar{J} + \partial x F^L E_0^{-1} E\bar{J} + JEE_0^{-1}F^R\bar{\partial}x\right. \\ &+ \left.\partial x(F - F^L E_0^{-1}F^R + F^L E_0^{-1}EE_0^{-1}F^R)\bar{\partial}x\right], \end{aligned} \tag{6.6}$$

or on $\tilde{G}$:

$$\tilde{\mathcal{S}} = \int d^2z\ \left[\left(\tilde{J} - \partial x F^L\right)\tilde{E}\left(\bar{\tilde{J}} + F^R\bar{\partial}x\right) + \partial x F\bar{\partial}x\right], \tag{6.7}$$

where $J = g^{-1}\partial g$ and $\tilde{J} = \tilde{g}^{-1}\partial\tilde{g}$.

Buscher-type rules that generalize the known rules for abelian [9] and traditional non-abelian [4, 5] dualities can be deduced from the above equations[6]:

$$\begin{aligned} &\left(E^{-1} - \mu\nu\right)^{-1} = \tilde{E}^{-1} - \beta\alpha = E^0(x^\alpha) \\ &E^0E^{-1}\mathcal{F}^R = \tilde{E}^{-1}\tilde{\mathcal{F}}^R = F^R(x^\alpha) \\ &\mathcal{F}^L E^{-1}E^0 = -\tilde{\mathcal{F}}^L\tilde{E}^{-1} = F^L(x^\alpha) \\ &\mathcal{F} + \mathcal{F}^L E^{-1}\left(E^0E^{-1} - \mathbf{1}\right)\mathcal{F}^R = \tilde{\mathcal{F}} - \tilde{\mathcal{F}}^L\tilde{E}^{-1}\tilde{\mathcal{F}}^R = F(x^\alpha) \\ &\Phi + \ln\det E^0 - \ln\det E = \tilde{\Phi} - \ln\det\tilde{E} = \Phi^0(x^\alpha)\,, \end{aligned} \tag{6.8}$$

where we have rewritten (6.6) as

$$\mathcal{S} = \int d^2z\ \left[JE\bar{J} + \partial x\mathcal{F}^L\bar{J} + J\mathcal{F}^R\bar{\partial}x + \partial x\mathcal{F}\bar{\partial}x\right] \tag{6.9}$$

[6]The formulas for general Poisson-Lie dualizeable backgrounds (as well as the corresponding Buscher rules) were first given in [1]. However, by using currents that transform under the action of the group instead of using invariant currents, these formulas, although correct, differ from the formulas of standard non-abelian duality [4, 5] by a similarity transformation, as was explained in section 3.

and introduced $\tilde{\mathcal{F}}^R$, $\tilde{\mathcal{F}}^L$, $\tilde{\mathcal{F}}$ similarly for (6.7). The method for applying Poisson-Lie T-duality can now be stated:

1. Look for a set of vector fields v_R and a set of constants $\tilde{f}^{ab}_c$ such that equations (6.3) are satisfied for the σ-model background. Notice that this is the generalization of what one usually does in ordinary duality where one looks for v's that satisfy the Killing equations $\mathcal{L}_v F = 0$.

2. The f's and the $\tilde{f}$'s specify the Drinfeld double used to calculate the matrices α, β of (2.16).

3. Use the formulas (6.8) to find the dual background.

We can check the result by analyzing the case when $\tilde{G}$ is abelian. In that case $E(g) = E_0$ and (6.6) reduces to a σ-model with non-abelian isometries and spectators. The form of the dual action (6.7) is unchanged and is the same as the one given in [4, 5]. We also see that equations (6.3) are satisfied for these backgrounds which means that they indeed satisfy non-abelian conservation laws.

7 Conclusions

We have presented the unambiguous path-integral derivation of Poisson-Lie duality and the most general actions that can be related by such duality (c.f. (6.6), (6.7)) leading to generalized Buscher rules (6.8). We have analyzed quantum determinants and discovered the non-trivial extra dilaton shifts (5.1), (5.2) that are needed to ensure quantum equivalence of the theories up to the one-loop order. Since difficulties in going beyond one loop exist already in the simplest case of abelian duality, we do not expect to do better here. The trace anomaly of traditional non-abelian duality for non-semisimple groups (whose structure constants are not traceless) is present in our case as well. With this in mind, one might look for non-trivial, *conformal* examples of Poisson-Lie T-duality. If we contract (2.10), assuming the structure constants of both groups to be traceless, we get a relation $f^c_{ab}\tilde{f}^{ab}_d = 0$ which tells us that such examples of Poisson-Lie T-duality can be found only for groups of dimension $D \geq 5$.

Our method is based on the ability to decompose any Drinfeld double element as a product of two group elements: $l = \tilde{h}g = h\tilde{g}$. As we have already noted, this can be done only close to the identity. It would be

interesting to investigate if there are any non-trivial effects associated with this fact.

It would also be interesting to perform a calculation similar to [7] in the Poisson-Lie case. We intend to address this problem in the future.

Note added: After this paper appeared on hep-th, a paper on related issues was submitted to hep-th [12]. There the authors find that our field $\tilde{A}$ in (4.6) (or A in (4.7)) has to obey an interesting non-local and non-linear "unit monodromy constraint"

$$P\exp\int_\gamma \tilde{A} = \tilde{e} \qquad\qquad \tilde{A} = \tilde{h}^{-1}\partial\tilde{h}, \tag{7.1}$$

where P stands for path-ordered exponential and γ is a closed path around the string world-sheet. This should give rise to non-perturbative effects in Poisson-Lie T-duality. However, in the present article we are only interested in perturbative results. Indeed, as stated above, our results are valid only to first order in perturbation theory. The uneasy reader may imagine the field $\tilde{A}$ as being expanded around a backgound which satisfies the constraint (7.1) leaving an unconstrained integration over infinitesimal fluctuations around the background. Of course, the background field drops out of the calculation.

Another way to look at this problem is to observe that (7.1) is just the well-known identity stating that a pure gauge field is flat

$$P\exp\int_\gamma \tilde{h}^{-1}d\tilde{h} \equiv \tilde{e}, \tag{7.2}$$

but written in light cone gauge. Thus the obstruction to making our statements globally valid is the same as the obstruction to imposing light cone gauge globally. In particular, this should not be a problem on any contractible patch on the world-sheet.

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References

[1] C. Klimčik, P. Ševera *"Dual Non-Abelian Duality and the Drinfeld Double"*, Phys. Lett. **B351** (1995) 455 or hep-th/**9502122**
C. Klimčik *"Poisson-Lie T-duality"*, hep-th/**9509095**

[2] A. Alekseev, C. Klimčik, A. Tseytlin *"Quantum Poisson-Lie T-duality and WZNW model"*, hep-th/**9509123**

[3] E. Tyurin *"Non-Abelian Axial-Vector Duality: a Geometric Description"*, hep-th/**9507014**

[4] F. Quevedo *"Abelian and Non-Abelian Dualities in String Backgrounds"*, hep-th/**9305055**
X. de la Ossa, F. Quevedo *"Duality Symmetries from Non-Abelian Isometries in String Theory"*, Nucl. Phys. **B403** (1993) 377 or hep-th/**9210021**

[5] A. Giveon, M. Roček *"On Nonabelian Duality"*, Nucl. Phys. **B421** (1994) 173 or hep-th/**9308154**

[6] E. Álvarez, L. Álvarez-Gaumè, Y. Lozano *"On Non-Abelian Duality"*, Nucl. Phys. **B424** (1994) 155 or hep-th/**9403155**

[7] E. Tyurin *"On Conformal Properties of the Dualized Sigma-Models"*, Phys. Lett. **B349** (1995) 386 or hep-th/**9411242**

[8] M. Roček , E. Verlinde *"Duality, Quotients and Currents"*, Nucl. Phys. **B373** (1992) 630 or hep-th/**9110053**

[9] T. Buscher *"Studies of the Two-dimensional Nonlinear Sigma-model"*, Ph. D. thesis (1988), unpublished
T. Buscher *"Path-Integral Derivation of Quantum Duality In Nonlinear Sigma-Models"*, Phys. Lett. **B201** (1988) 466
T. Buscher *"A Symmetry of the String Background Field Equations"*, Phys. Lett. **B194** (1987) 59

[10] A. Alekseev, A. Malkin *"Symplectic Structures Associated to Lie-Poisson Groups"* Comm. Math. Phys. **162** (1994) 147

[11] A. Schwarz, A. Tseytlin *"Dilaton Shift under Duality and Torsion of Elliptic Complex"* Nucl. Phys. **B399** (1993) 691 or hep-th/**9210015**

[12] C. Klimčik, P. Ševera *"Poisson-Lie T-duality and Loop Groups of Drinfeld Doubles"* hep-th/**9512040**